

Chaotic Transport in the Symmetry Crossover Regime with a Spin-orbit Interaction

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We study a chaotic quantum transport in the presence of a weak spin-orbit interaction. Our theory covers the whole symmetry crossover regime between time-reversal invariant systems with and without a spin-orbit interaction. This situation is experimentally realizable when the spin-orbit interaction is controlled in a conductor by applying an electric field. We utilize a semiclassical approach which has recently been developed. In this approach, the non-Abelian nature of the spin diffusion along a classical trajectory plays a crucial role. New analytical expressions with one crossover parameter are semiclassically derived for the average conductance, conductance variance and shot noise. Moreover numerical results on a random matrix model describing the crossover from the GOE (Gaussian Orthogonal Ensemble) to the GSE (Gaussian Symplectic Ensemble) are compared with the semiclassical expressions.

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I. INTRODUCTION

A chaotic quantum transport of an electron in a cavity is caused by either implanted impurities or bumpy boundaries, and provides directly measurable *quantum signatures of chaos*¹, such as the conductance variance. Universal aspects of a chaotic transport have been investigated by means of the random matrix theory (RMT)². In the RMT, quantum systems are classified into symmetry classes. A chaotic system with time-reversal symmetry is described by the Gaussian orthogonal ensemble (GOE). When the time-reversal symmetry is broken by applying a magnetic field, the Gaussian unitary ensemble (GUE) becomes a suitable model. If a system with time-reversal symmetry has a spin-orbit interaction, one needs to employ the Gaussian symplectic ensemble (GSE).

We consider the case that two leads are attached to a cavity and the number of the lead channels are N_1 and N_2 . An electron transport in the cavity is described by the scattering matrix^{3,4}. Replacing the scattering matrix by a random matrix, the RMT phenomenologically predicts the average conductance G , conductance variance $\text{Var}G$, and shot noise P at zero temperature as^{5,6}

$$\begin{aligned} \frac{G}{G_0} &= \frac{2N_1N_2}{N-1+\frac{2}{\beta}} \\ &= \frac{2N_1N_2}{N} \left\{ 1 + \frac{1-\frac{2}{\beta}}{N} + \left(\frac{1-\frac{2}{\beta}}{N} \right)^2 \right\} \\ &\quad + O\left(\frac{1}{N^2}\right), \end{aligned} \quad (1)$$

$$\begin{aligned} \frac{\text{Var}G}{G_0^2} &= \frac{8N_1(N_1-1+\frac{2}{\beta})N_2(N_2-1+\frac{2}{\beta})}{\beta(N-2+\frac{2}{\beta})(N-1+\frac{2}{\beta})^2(N-1+\frac{4}{\beta})} \\ &= \frac{8N_1^2N_2^2}{\beta N^4} + O\left(\frac{1}{N}\right), \end{aligned} \quad (2)$$

$$\begin{aligned} \frac{P}{P_0} &= \frac{2N_1(N_1-1+\frac{2}{\beta})N_2(N_2-1+\frac{2}{\beta})}{(N-2+\frac{2}{\beta})(N-1+\frac{2}{\beta})(N-1+\frac{4}{\beta})} \\ &= \frac{2N_1^2N_2^2}{N^3} + \left(\frac{4}{\beta}-2\right) \frac{N_1N_2}{N^4} (N_1-N_2)^2 \\ &\quad + O\left(\frac{1}{N}\right) \end{aligned} \quad (3)$$

with $N = N_1 + N_2$. Here $\beta = 1, 2$, and 4 correspond to the GOE, GUE, and GSE symmetry classes, respectively. These expressions include the contributions from the spin degrees of freedom⁷. The constants G_0 and P_0 are $G_0 = e^2/(\pi\hbar)$ and $P_0 = 2e^3|V|/(\pi\hbar)$, respectively, where e is the unit electric charge and V is the bias voltage. If N_1 is equal to N_2 and very large, the leading term of the shot noise is insensitive to a change of the symmetry.

When a very weak magnetic field is applied to the cavity, the time-reversal symmetry is only partially broken. In this case, a crossover from the GOE to GUE is observed. This GOE-GUE crossover regime can also be analyzed by a parametric RMT model, and analytic predictions describing a chaotic quantum transport are known⁸. Recently, a chaotic transport in the GSE-GUE crossover regime was also studied within the RMT framework⁹. In this regime, a very weak magnetic field breaks the time-reversal symmetry of a system with a spin-orbit interaction.

The aim of this paper is to study another case, the crossover from the GOE to GSE, in which the system has a very weak spin-orbit interaction preserving the time reversal symmetry. In the experimental point of view, the GOE-GSE crossover can be realized, if the spin-orbit interaction (or Aharonov-Casher effect) is controlled by applying an electric field in a chaotic conductor^{10,11}. In this case, a parametric RMT model is also known^{12,13}, and the diagrammatic perturbation theory has been used to evaluate some transport properties¹⁴⁻¹⁷. Here we employ a semiclassical approach which has recently

been developed^{18–23}. In a semiclassical evaluation, the transmission amplitude is treated by the path-integral method, where all the classical paths must in principle be taken into account. However recent studies clarified that almost the same but partially time reversed pairs of classical trajectories contributed to the conductance^{24–27}, so that the calculation was greatly simplified. Then it was shown that the semiclassical approach could precisely reproduce the RMT predictions (1)-(3)^{21–23}. Moreover, when a similar approach is applied to the parametric spectral correlations in the GOE-GUE, GUE-GUE, GOE-GOE and GSE-GSE regimes, it can also reproduce the RMT predictions^{28–31}.

Thus the semiclassical approach has become a practical tool to find a new prediction, even if the RMT analysis is difficult. As this approach was already applied to the parametric spectral correlations in the GOE-GSE crossover regime³¹, we naturally expect that it can be used in the analysis of a transport.

Considering the non-Abelian nature of the spin diffusion along the classical trajectories, we extend the semiclassical technique to derive analytic expressions for the transport properties. Our results on the average conductance, conductance variance and shot noise are given in eqs. (25), (31), and (35). The crossover from the GOE to GSE is controlled by one parameter depending on the diffusion constant of the spin. The GOE and GSE results are reproduced in the limiting cases of the parameter.

This paper is organized in the following way. In Sec. II, a semiclassical expression of the transmission amplitude is presented. We put a stress on the statistical aspects of the expression. In Sec. III, using the semiclassical expression, we calculate the average conductance, conductance variance and shot noise. In Sec. IV, these results are compared with numerical calculations on a random matrix model. We finally summarize the paper in Sec. V.

II. SEMICLASSICAL EXPRESSION OF THE TRANSMISSION AMPLITUDE

The semiclassical theory employs the transmission amplitude t_{a_1, a_2} , which represents the propagator of a wave packet from the channel a_1 in one lead to a_2 in another lead. Bolte and Keppeler derived a semiclassical expression of the transmission amplitude with spin variables³²:

$$t_{a_1, a_2} \sim \sqrt{\frac{2}{T_H}} \sum_{\alpha: a_1 \rightarrow a_2} \mathcal{A}_\alpha \Delta_\alpha e^{iS_\alpha/\hbar}, \quad (4)$$

where T_H is the Heisenberg time $T_H = \frac{\Omega(E)}{(2\pi\hbar)^{f-1}}$. Here $\Omega(E)$ is the phase volume density including spin degrees of freedom at the energy E , and f is the spacial dimension. Throughout this paper, we study the two dimensional case $f = 2$. Two leads are assumed to have N_1 and N_2 channels, i.e. $a_1 = 1, 2, \dots, N_1$ and $a_2 = 1, 2, \dots, N_2$.

The classical action of the orbit α is $S_\alpha = \int_\alpha \mathbf{p} \cdot d\mathbf{q}$, where \mathbf{q} and \mathbf{p} are the position and momentum variables.

The stability amplitude is decoupled into two factors \mathcal{A}_α and Δ_α . The first factor \mathcal{A}_α accounts for the stability in the position and momentum space, and the second factor Δ_α originates from the spin dynamics. Both \mathcal{A}_α and Δ_α are uniquely determined, when the classical trajectory α governed by the microscopic Hamiltonian is specified. However their statistical behavior is independent of the details of the trajectory. As discussed in Refs.21,22 and 33, the stability amplitude \mathcal{A}_α is related to the survival probability in the chaotic cavity. Postulating an ergodic motion in the position and momentum space, we obtain the following sum rule

$$\sum_{\alpha: a_1 \rightarrow a_2} |\mathcal{A}_\alpha|^2 = \int_0^\infty dT e^{-(2N/T_H)T}, \quad (5)$$

where $T_H/2N$ ($N = N_1 + N_2$) can be regarded as the dwell time inside the cavity, so that the inverse is the escape rate. The escape rate is related to the position and momentum variables and is unrelated to the spin variables. Hence the dwell time should be $T_H/2N$ rather than T_H/N . The spin matrix Δ_α is defined as

$$\Delta(t) = e^{i\phi(t)\sigma_z/2} e^{i\theta(t)\sigma_x/2} e^{i\psi(t)\sigma_z/2} \quad (6)$$

along a trajectory α , where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ consists of the Pauli matrices. The time evolution of the Euler angles $(\psi(t), \theta(t), \phi(t))$ is microscopically determined by the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \Delta(t) = \mathcal{H} \Delta(t), \quad (7)$$

where \mathcal{H} is the effective Hamiltonian which describes the spin-orbit interaction. We assume that the spin dynamics is subdominant in the semiclassical limit. That is, the dynamics of the position and momentum variables are determined by the spacial Hamiltonian without spin degrees of freedom, while the spin is influenced by the momentum motion via the spin-orbit interaction. The effective Hamiltonian in (7) describes such a subdominant dynamics of the spin variables. A similar hierarchical structure has successfully been employed to analyze the GOE-GUE crossover regime^{22,28–30}: the resulting physical quantities are in agreement with the corresponding RMT expressions. An RMT prediction for the parametric spectral form factor in the GSE symmetry class was also reproduced in a similar way³¹.

Since bumpy boundaries of a cavity induce a chaotic behavior in the position and momentum variables, the momentum effectively plays a role of a stochastic magnetic field³¹. Then the effective Hamiltonian which describes the time evolution of the spin can be written as

$$\mathcal{H} = \gamma_{\text{so}} \mathbf{h} \cdot \left(\frac{\hbar}{2} \boldsymbol{\sigma} \right), \quad (8)$$

where γ_{so} is the coupling constant and $\mathbf{h} = (h_x(t), h_y(t), h_z(t))$ is the effective stochastic magnetic

field. We assume that the classical spin undergoes a Brownian motion on the Bloch sphere³⁴ due to the stochastic magnetic field satisfying

$$\langle\langle h_\alpha(t)h_{\alpha'}(t') \rangle\rangle = 2\mathcal{D}\delta_{\alpha,\alpha'}\delta(t-t'), \quad \alpha, \alpha' = x, y, z, \quad (9)$$

where the brackets $\langle\langle \dots \rangle\rangle$ denote an average over the stochastic process of the magnetic field and \mathcal{D} is the diffusion constant. Then the probability density function of the Euler angle $P(\psi, \theta, \phi)$ (with the measure $\sin\theta d\psi d\theta d\phi$) obeys the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \gamma_{\text{so}}^2 \mathcal{D} \mathcal{L} P, \quad (10)$$

where \mathcal{L} is the Laplace-Beltrami operator

$$\begin{aligned} \mathcal{L} = & \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} \\ & + \frac{1}{\sin^2\theta} \left(\frac{\partial^2}{\partial\psi^2} + \frac{\partial^2}{\partial\phi^2} - 2\cos\theta \frac{\partial^2}{\partial\psi\partial\phi} \right). \end{aligned} \quad (11)$$

This stochastic dynamics can exactly be analyzed by using Wigner's D function³⁵. Let us suppose that the initial Euler angles at $t = 0$ are $\omega' = (\psi', \theta', \phi')$. Then the conditional probability to find the angles at $\omega = (\psi, \theta, \phi)$ after time t is

$$\begin{aligned} g(\omega; t|\omega') = & \sum_{j=0}^{\infty} \sum_{m=-j}^j \sum_{n=-j}^j \frac{2j+1}{32\pi^2} \\ & \times D_{m,n}^j(\psi, \theta, \phi) \{D_{m,n}^j(\psi', \theta', \phi')\}^* e^{-j(j+1)\gamma_{\text{so}}^2 \mathcal{D} t}. \end{aligned} \quad (12)$$

Here an asterisk signifies a complex conjugate and Wigner's D function is defined as

$$D_{m,n}^j(\psi, \theta, \phi) = e^{im\phi} d_{m,n}^j(\theta) e^{in\psi}, \quad (13)$$

where

$$\begin{aligned} d_{m,n}^j(\theta) = & \sqrt{\frac{(j+m)!(j-m)!}{(j+n)!(j-n)!}} \\ & \times \cos^{m+n}(\theta/2) \sin^{m-n}(\theta/2) P_{j-m}^{(m-n, m+n)}(\cos\theta) \end{aligned} \quad (14)$$

in terms of the Jacobi polynomials $P_k^{(a,b)}(x)$. The index j is an integer or a half odd integer ($j = 0, 1/2, 1, 3/2, \dots$ and $m, n = -j, -j+1, \dots, j$). The conditional probability $g(\omega; t|\omega')$ satisfies a normalization condition

$$\int d\omega g(\omega; t|\omega') = \int_0^\pi d\theta \int_0^{4\pi} d\phi \int_0^{4\pi} d\psi \sin\theta g(\omega; t|\omega') = 1.$$

III. TRANSPORT IN THE GOE-GSE CROSSOVER REGIME

A. Average conductance

The average conductance G is written in terms of the transmission amplitude t_{a_1, a_2} as

$$\frac{G}{G_0} = \langle \text{Tr}(\mathbf{t}\mathbf{t}^\dagger) \rangle = \sum_{a_1=1}^{N_1} \sum_{a_2=1}^{N_2} \text{Tr}\{t_{a_1, a_2}(t^\dagger)_{a_2, a_1}\}. \quad (15)$$

Here the transmission matrix \mathbf{t} is a $2N_1 \times 2N_2$ matrix which consists of the 2×2 blocks t_{a_1, a_2} . Then a semiclassical expression

$$\begin{aligned} \langle \text{Tr}(\mathbf{t}\mathbf{t}^\dagger) \rangle = & \frac{2}{T_H} \left\langle \sum_{a_1, a_2} \sum_{\alpha, \gamma: a_1 \rightarrow a_2} \mathcal{A}_\alpha \mathcal{A}_\gamma^* \langle \text{Tr}(\Delta_\alpha \Delta_\gamma^\dagger) \rangle e^{\frac{i}{\hbar}(S_\alpha - S_\gamma)} \right\rangle \end{aligned} \quad (16)$$

follows from (4). Here the brackets $\langle \dots \rangle$ mean an energy average, which eliminates the fluctuations of the physical quantities. If the difference between the actions S_α and S_γ is sufficiently large, the exponential term $e^{\frac{i}{\hbar}(S_\alpha - S_\gamma)}$ rapidly oscillates in the semiclassical limit $\hbar \rightarrow 0$, which eventually vanishes after averaging. Hence, in order to give a finite contribution, the trajectories α and γ are mutually almost the same. Then the identical trajectories $\alpha = \gamma$ yield the first order approximation, which is referred to as “the diagonal approximation”³⁶. These mutually identical trajectory pairs yield the following contribution

$$\begin{aligned} \langle \text{Tr}(\mathbf{t}\mathbf{t}^\dagger) \rangle_1 = & \frac{2}{T_H} \sum_{a_1, a_2} \sum_{\alpha} \text{Tr}(\Delta_\alpha \Delta_\alpha^\dagger) |\mathcal{A}_\alpha|^2 \\ = & \frac{4}{T_H} \sum_{a_1, a_2} \sum_{\alpha} |\mathcal{A}_\alpha|^2 \\ = & \frac{4}{T_H} N_1 N_2 \int_0^\infty dT e^{-(2N/T_H)T} \\ = & \frac{2N_1 N_2}{N}. \end{aligned} \quad (17)$$

Here we used the sum rule (5) for the stability amplitude \mathcal{A}_α . In this calculation, a product of the spin matrices is reduced to the identity matrix, and the trace yields a factor 2. The diagonal approximation does not discriminate the symmetry classes.

The second order approximation comes from the Richter-Sieber (RS) pairs^{26,27} drawn in Fig.1. In the RS pair, two trajectories come close to each other in the encounter region, and go in the opposite directions on one loop. We can symbolically write RS pairs (See Fig.1) as

$$\begin{aligned} \alpha : & L_1 E L_2 \bar{E} L_3, \\ \gamma : & L_1 E \bar{L}_2 \bar{E} L_3, \end{aligned}$$

where E implies one of the two trajectory segments in the encounter region where two loops are connected and \bar{E} implies the time reverse of E . The loops are denoted as L_1 , L_2 and L_3 , respectively, and \bar{L}_j ($j = 1, 2, 3$) is the time reverse of L_j . Using these notations, we can write the RS pair contribution as

$$\begin{aligned} & \langle \text{Tr}(\mathbf{t}\mathbf{t}^\dagger) \rangle_2 \\ &= \frac{2}{T_H} \sum_{a_1, a_2} \sum_{\substack{\alpha: L_1 E L_2 \bar{E} L_3 \\ \gamma: L_1 \bar{E} L_2 E L_3}} \left\langle \mathcal{A}_\alpha \mathcal{A}_\gamma^* \langle \langle \text{Tr}(\Delta_\alpha \Delta_\gamma^\dagger) \rangle \rangle e^{\frac{i}{\hbar} \Delta S} \right\rangle, \end{aligned} \quad (18)$$

where ΔS is the action difference $S_\alpha - S_\gamma$. The spin matrices Δ_α and Δ_γ are factored into the loop and encounter parts as

$$\Delta_\alpha = \Delta_{L_1} \Delta_E \Delta_{L_2} (\Delta_E)^{-1} \Delta_{L_3}, \quad (19)$$

$$\Delta_\gamma = \Delta_{L_1} \Delta_E (\Delta_{L_2})^{-1} (\Delta_E)^{-1} \Delta_{L_3}. \quad (20)$$

Along the trajectories the non-Abelian nature of the spin operators must be taken into account. A time reversal operation of a spin matrix is realized by a matrix inversion.

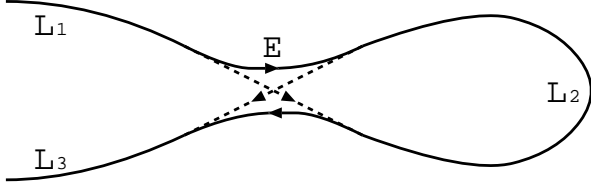


FIG. 1: The Richter-Sieber (RS) pair. The solid and dashed curves are respectively α and γ orbits in the text.

We divide the whole time elapsed on a trajectory into the loop and encounter parts, i.e., T_1, T_2, T_3 for L_1, L_2, L_3 , respectively, and t_{enc} for E . It should be noted that the existence of the encounter affects the survival probability $e^{-(2N/T_H)T}$ in Eq. (5)^{21,22}: if one of the two orbit segments in the encounter is inside the cavity, the other segment must also remain inside. Hence the survival probability is modified into $e^{-(2N/T_H)(T-t_{\text{enc}})}$.

In the encounter region, the classical actions of the two trajectories are slightly different. The action difference can be estimated by using the coordinates (s, u) along the stable and unstable manifolds within the ranges $s, u \in [-c, c]$ ^{21,22}. The time duration t_{enc} inside the encounter region is related to the Lyapunov exponent λ as $t_{\text{enc}} \sim \frac{1}{\lambda} \ln \frac{c^2}{|us|}$, and the action difference is expressed as

$$\Delta S = us. \quad (21)$$

The number density of encounters in a trajectory with an elapsed time $T = T_1 + T_2 + T_3 + 2t_{\text{enc}}$ is evaluated

as^{21,22}

$$\omega(s, u) ds du = \int_{\substack{T_1, T_2 > 0 \\ T_1 + T_2 < T - 2t_{\text{enc}}}} dT_1 dT_2 \frac{1}{t_{\text{enc}} \Omega / 2} ds du \quad (22)$$

by taking account of $T_1, T_2 > 0$ and $T_3 = T - T_1 - T_2 - 2t_{\text{enc}} > 0$.

On the other hand, the spin diffusion term is calculated as

$$\begin{aligned} \langle \langle \text{Tr}(\Delta_\alpha \Delta_\gamma^\dagger) \rangle \rangle &= \int d\omega_{L_1} d\omega_{L_2} d\omega_{L_3} d\omega_{E_1} \text{Tr} \{ (\Delta_{L_2})^2 \} \\ &\times g(\omega_{L_1}, T_1 | 0, 0, 0) g(\omega_{L_2}, T_2 | 0, 0, 0) \\ &\times g(\omega_{L_3}, T_3 | 0, 0, 0) g(\omega_{E_1}, t_{\text{enc}} | 0, 0, 0) \\ &= -1 + 3e^{-2\gamma_{\text{so}}^2 \mathcal{D} T_2}. \end{aligned} \quad (23)$$

Using the above formulas, we find that the RS contribution is

$$\begin{aligned} \langle \text{Tr}(\mathbf{t}\mathbf{t}^\dagger) \rangle_2 &= \frac{2N_1 N_2}{T_H} \int_{-c}^c ds du \int_{t_{\text{enc}}}^\infty dT \\ &\times \omega(s, u) e^{-(2N/T_H)(T-t_{\text{enc}})} \langle \langle \text{Tr}(\Delta_\alpha \Delta_\gamma^\dagger) \rangle \rangle \\ &= \frac{2N_1 N_2}{T_H} \int_0^\infty dT_1 dT_2 dT_3 \int_{-c}^c ds du \frac{2e^{isu/\hbar}}{\Omega t_{\text{enc}}} \\ &\times e^{-(2N/T_H)(T_1+T_2+T_3+t_{\text{enc}})} (3e^{-2\gamma_{\text{so}}^2 \mathcal{D} T_2} - 1) \\ &= \frac{N_1 N_2}{(N_1 + N_2)^2} \left(1 - \frac{3}{1 + \gamma_{\text{so}}^2 \mathcal{D} T_H / N} \right). \end{aligned} \quad (24)$$

The last line of the above formula is obtained by the following criterion: after expanding the formula in t_{enc} and integrating each term over (s, u) , any term dependent on t_{enc} vanishes in the semiclassical limit, and a finite contribution comes only from the terms independent of t_{enc} .

The third order contribution to the average conductance comes from the diagrams drawn in Fig.2. In Appendix A, the spin diffusion terms are listed. Using these results, we arrive at an expression of the average conductance

$$\begin{aligned} \frac{G}{G_0} &= \frac{2N_1 N_2}{N} + \frac{N_1 N_2}{N^2} \left(1 - \frac{3}{1 + \xi} \right) \\ &+ \frac{N_1 N_2}{2N^3} \left\{ 1 - \frac{3}{1 + \xi} + \frac{3}{(1 + \xi)^2} + \frac{3}{(1 + \xi)^3} \right\} \\ &+ O(1/N^2), \end{aligned} \quad (25)$$

where ξ is the crossover parameter defined as $\xi = \gamma_{\text{so}}^2 \mathcal{D} T_H / N$. We can easily check that (25) reproduces the $1/N$ expansions of the GOE and GSE formulas (1) by taking the limits $\xi \rightarrow 0$ and $\xi \rightarrow \infty$, respectively. It follows from this result that only one parameter ξ is necessary to describe the GOE-GSE crossover.

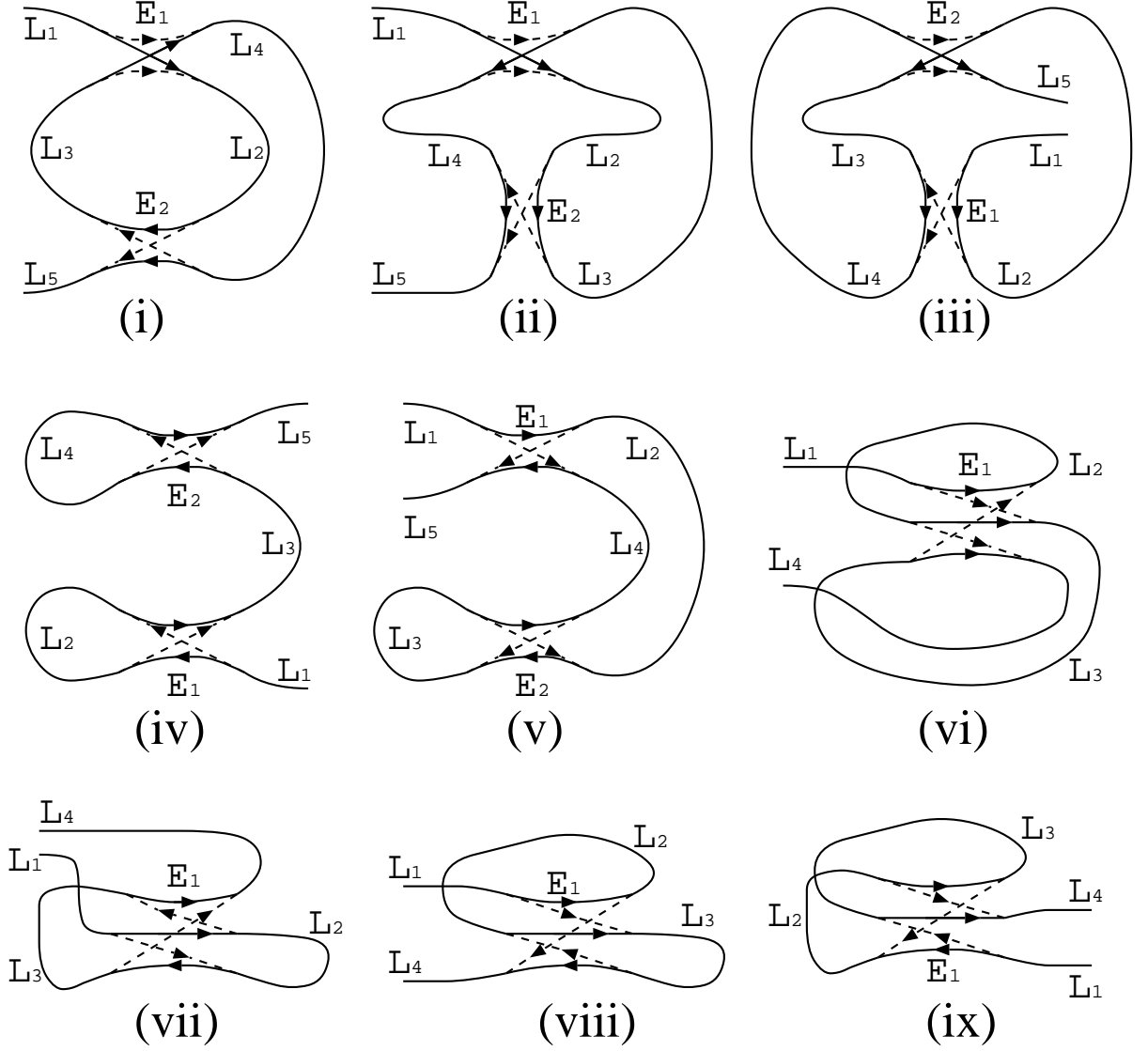


FIG. 2: The diagrams of the third order contributing to the conductance

B. Conductance variance

The conductance variance $\text{Var}G$ is written in terms of the transmission matrix \mathbf{t} as

$$\frac{\text{Var}G}{G_0^2} = \langle \{ \text{Tr}(\mathbf{t}\mathbf{t}^\dagger) \}^2 \rangle - \langle \text{Tr}(\mathbf{t}\mathbf{t}^\dagger) \rangle^2. \quad (26)$$

The semiclassical expression of the first term is

$$\begin{aligned} \langle \{ \text{Tr}(\mathbf{t}\mathbf{t}^\dagger) \}^2 \rangle &= \frac{4}{T_H^2} \sum_{a_1, a_2} \sum_{c_1, c_2} \sum_{\alpha, \beta: a_1 \rightarrow a_2} \sum_{\gamma, \delta: c_1 \rightarrow c_2} \\ &\times \langle \mathcal{A}_\alpha \mathcal{A}_\beta^* \mathcal{A}_\gamma \mathcal{A}_\delta^* e^{i(S_\alpha - S_\beta + S_\gamma - S_\delta)/\hbar} \\ &\times \langle \langle \text{Tr}(\Delta_\alpha \Delta_\beta^\dagger) \text{Tr}(\Delta_\gamma \Delta_\delta^\dagger) \rangle \rangle \rangle. \end{aligned} \quad (27)$$

Let us first adopt the diagonal approximation, for which we have two types of contributions. One is given by setting $\alpha = \beta$ and $\gamma = \delta$, where a_1, a_2, c_1 and c_2 are all independent. The other choice is $\alpha = \delta, \gamma = \beta$, and $a_1 = c_1, a_2 = c_2$. The numbers of possible channel combinations are $N_1^2 N_2^2$ and $N_1 N_2$, respectively. These contributions are summed up to yield

$$\begin{aligned} \langle \{ \text{Tr}(\mathbf{t}\mathbf{t}^\dagger) \}^2 \rangle_1 &= \frac{4}{T_H^2} \left\langle 4 \sum_{\substack{a_1, c_1 \\ a_2, c_2}} \sum_{\substack{\alpha = \beta: a_1 \rightarrow a_2 \\ \gamma = \delta: c_1 \rightarrow c_2}} |\mathcal{A}_\alpha|^2 |\mathcal{A}_\gamma|^2 \right. \\ &+ \left. \sum_{\substack{a_1 = c_1 \\ a_2 = c_2}} \sum_{\substack{\alpha = \delta: a_1 \rightarrow a_2 \\ \gamma = \beta: a_1 \rightarrow a_2}} |\mathcal{A}_\alpha|^2 |\mathcal{A}_\gamma|^2 \langle \langle \text{Tr}(\Delta_\alpha \Delta_\gamma^\dagger) \rangle \rangle \right\rangle. \end{aligned} \quad (28)$$

Here the spin diffusion term is

$$\begin{aligned}
& \langle |\text{Tr}(\Delta_\alpha \Delta_\gamma^\dagger)|^2 \rangle \\
&= \int d\omega_\alpha d\omega_\gamma g(\omega_\alpha, T_\alpha | 0, 0, 0) g(\omega_\gamma, T_\gamma | 0, 0, 0) \\
&= 1 + 3e^{-2\gamma_{\text{so}}^2 \mathcal{D}(T_\alpha + T_\gamma)}, \tag{29}
\end{aligned}$$

where T_α and T_γ are the times elapsed on the trajectories α and γ , respectively. Using this expression, one can

obtain the diagonal contribution

$$\left\langle \left\{ \text{Tr}(\mathbf{t}\mathbf{t}^\dagger) \right\}^2 \right\rangle_1 = \frac{4N_1^2 N_2^2}{N^2} + \frac{N_1 N_2}{N^2} \left\{ 1 + \frac{3}{(1+\xi)^2} \right\}. \tag{30}$$

To go beyond the diagram approximation, we note that the next order diagrams are classified into d -families and x -families as shown in Fig.3²²: d -quadruplets are drawn in the diagrams (i)-(vii), while x -quadruplets in (viii).

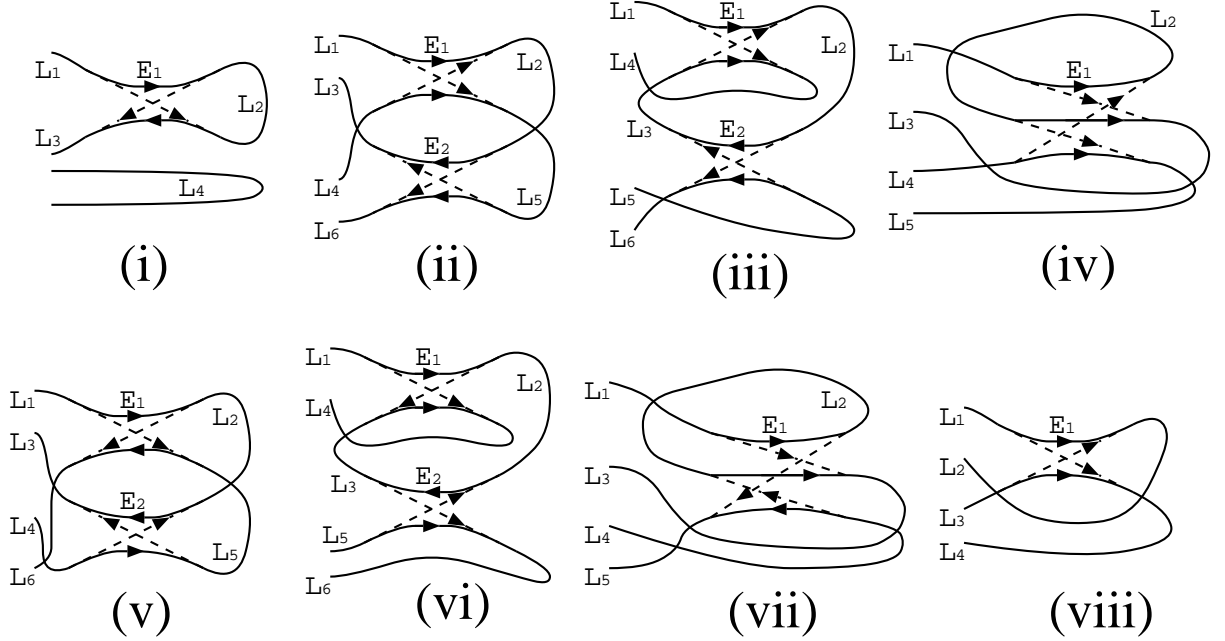


FIG. 3: The diagrams contributing to the conductance variance

Let us write the next order term as

$$\begin{aligned}
& \left\langle \left\{ \text{Tr}(\mathbf{t}\mathbf{t}^\dagger) \right\}^2 \right\rangle_2 \\
&= (N_1^2 N_2^2 + N_1 N_2) \left(\frac{d_1}{N^3} + \frac{d_2}{N^4} \right) + N_1 N_2 \frac{x_1}{N^2}.
\end{aligned}$$

Here the coefficients d_1 , d_2 and x_1 are obtained from the families of quadruplets. The coefficient d_1 comes from the diagram (i): quadruplets consisting of one diagonal pair and one RS pair. On the other hand, many diagrams have to be taken into account to calculate d_2 , i.e., 1): quadruplets consisting of one diagonal pair and one pair contributing to the $O(1/N)$ term in the expansion (25) of the average conductance, 2): two RS pairs 3): the diagrams (ii)-(vii) shown in Fig.3. The coefficient x_1 is calculated from the diagram (viii) in Fig.3. Considering

the contributions from these diagrams, we obtain

$$\begin{aligned}
d_1 &= 4 - \frac{12}{1+\xi}, \\
d_2 &= 5 - \frac{12}{1+\xi} + \frac{21}{(1+\xi)^2} + \frac{6}{(1+\xi)^3}, \\
x_1 &= -1 - \frac{3}{(1+\xi)^2}.
\end{aligned}$$

Then we find the expression of the conductance variance for $N_1 \gg 1$ and $N_2 \gg 1$ as

$$\frac{\text{Var}G}{G_0^2} = \frac{N_1^2 N_2^2}{N^4} \left\{ 2 + \frac{6}{(1+\xi)^2} \right\} + O\left(\frac{1}{N}\right). \tag{31}$$

One can check that the GOE and GSE limits agree with (2).

C. Shot noise

The shot noise P is written in the form

$$\frac{P}{P_0} = \langle \text{Tr}(\mathbf{t}\mathbf{t}^\dagger - \mathbf{t}\mathbf{t}^\dagger\mathbf{t}\mathbf{t}^\dagger) \rangle. \quad (32)$$

The second term is semiclassically expressed as

$$\begin{aligned} \langle \text{Tr}(\mathbf{t}\mathbf{t}^\dagger\mathbf{t}\mathbf{t}^\dagger) \rangle &= \frac{4}{T_H^2} \sum_{\substack{a_1, a_2 \\ c_1, c_2}} \sum_{\substack{\alpha: a_1 \rightarrow a_2, \beta: c_1 \rightarrow a_2 \\ \gamma: c_1 \rightarrow c_2, \delta: a_1 \rightarrow c_2}} \\ &\times \left\langle A_\alpha A_\beta^* A_\gamma A_\delta^* e^{\frac{i}{\hbar}(S_\alpha - S_\beta + S_\gamma - S_\delta)} \right. \\ &\times \left. \langle \langle \text{Tr}(\Delta_\alpha \Delta_\beta^\dagger \Delta_\gamma \Delta_\delta^\dagger) \rangle \rangle \right\rangle. \end{aligned} \quad (33)$$

The diagonal contribution consists of two terms. One term has $\alpha = \beta$ and $\gamma = \delta$ where we need to set $a_1 = c_1$. The other term has $\alpha = \delta$, $\beta = \gamma$, and $a_2 = c_2$. We sum up these two terms and obtain

$$\begin{aligned} &\langle \text{Tr}(\mathbf{t}\mathbf{t}^\dagger\mathbf{t}\mathbf{t}^\dagger) \rangle_1 \\ &= \frac{16}{T_H^2} \left\langle \sum_{a_1, a_2, c_2} \sum_{\substack{\alpha: a_1 \rightarrow a_2 \\ \gamma: a_1 \rightarrow c_2}} |A_\alpha|^2 |A_\gamma|^2 \langle \langle \text{Tr}(\Delta_\alpha \Delta_\alpha^\dagger \Delta_\gamma \Delta_\gamma^\dagger) \rangle \rangle \right. \\ &\quad \left. + \sum_{a_1, a_2, c_1} \sum_{\substack{\alpha: a_1 \rightarrow a_2 \\ \beta: c_1 \rightarrow a_2}} |A_\alpha|^2 |A_\beta|^2 \langle \langle \text{Tr}(\Delta_\alpha \Delta_\beta^\dagger \Delta_\beta \Delta_\alpha^\dagger) \rangle \rangle \right\rangle \\ &= 2 \frac{N_1 N_2}{N}. \end{aligned} \quad (34)$$

It follows that the diagonal contributions to $\langle \text{Tr}(\mathbf{t}\mathbf{t}^\dagger) \rangle$ and $\langle \text{Tr}(\mathbf{t}\mathbf{t}^\dagger\mathbf{t}\mathbf{t}^\dagger) \rangle$ are both $2 \frac{N_1 N_2}{N}$, and mutually cancel.

Hence the RS pair contribution to $\text{Tr}(\mathbf{t}\mathbf{t}^\dagger)$, and the contribution to $\text{Tr}(\mathbf{t}\mathbf{t}^\dagger\mathbf{t}\mathbf{t}^\dagger)$ from the quadruplets drawn in Fig.3 (i) and (viii) have to be calculated. Moreover we take account of the additional diagrams shown in Fig.4^{21,22}. Summing up these contributions, we finally obtain the shot noise in the crossover regime as

$$\begin{aligned} \frac{P}{P_0} &= 2 \frac{N_1^2 N_2^2}{N^3} + \frac{N_1 N_2 (N_1 - N_2)^2}{N^4} \left(\frac{3}{1 + \xi} - 1 \right) \\ &\quad + O(1/N). \end{aligned} \quad (35)$$

Let us denote the $O(1)$ terms of G and P by δG and δP , respectively. It can be seen from Eqs. (25) and (35) that

$$\frac{\delta P / P_0}{\delta G / G_0} = - \left(\frac{N_1 - N_2}{N_1 + N_2} \right)^2, \quad (36)$$

which is a universal relation established in Ref.17.

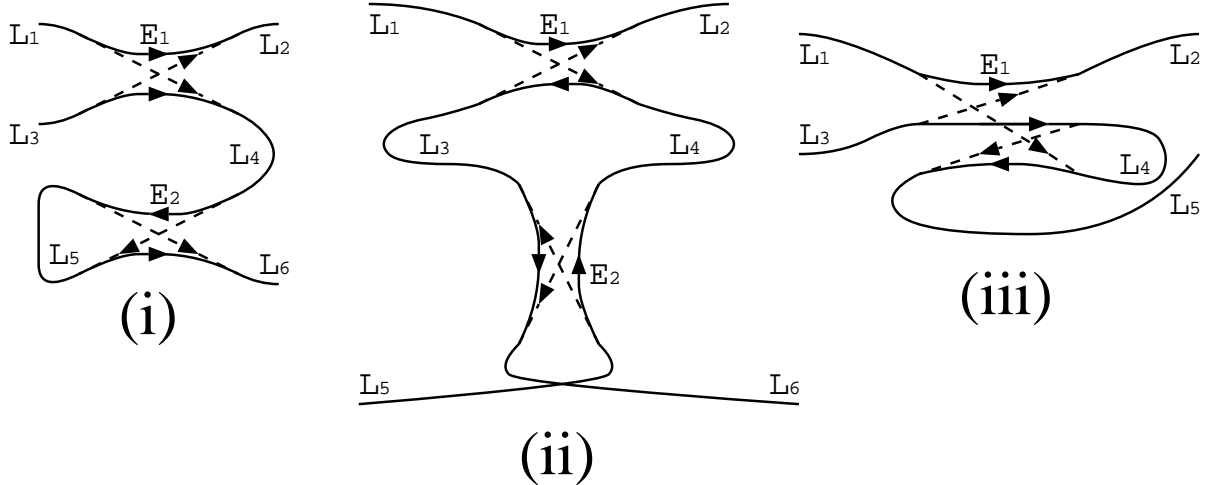


FIG. 4: The diagrams contributing to the shot noise

IV. COMPARISON WITH A RANDOM MATRIX MODEL

In this section, a random matrix model on the GOE-GSE crossover is numerically analyzed and the results are

compared with the semiclassical formulas. In the theory of random matrices, a time-reversal invariant quantum Hamiltonian with spin 1/2 is simulated by a self-dual

real quaternion random matrix³⁷. A real quaternion q is a linear combination

$$q = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3, \quad (37)$$

where q_j are real numbers and called the j -th component of q . The bases e_0, e_1, e_2, e_3 can be represented by 2×2 matrices as

$$\begin{aligned} e_0 &\rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & e_1 &\rightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \\ e_2 &\rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & e_3 &\rightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \end{aligned} \quad (38)$$

so that $q_0 e_0$ is equated with a real number q_0 . The dual quaternion of q is defined as

$$\bar{q} = q_0 e_0 - q_1 e_1 - q_2 e_2 - q_3 e_3. \quad (39)$$

When an $m \times n$ real quaternion matrix Q has the (j, l) element q_{jl} , we define that the $n \times m$ dual matrix \bar{Q} has the (j, l) element \bar{q}_{lj} . If a square real quaternion matrix satisfies $Q = \bar{Q}$, then Q is called a self-dual real quaternion matrix.

The parametric motion of a self-dual real quaternion random matrix is realized in the framework of Dyson's matrix Brownian motion model³⁸. It is postulated that the probability density function (p.d.f.) of an $M \times M$ self-dual real quaternion matrix H is

$$P(H; \tau | R) dH \propto \exp \left\{ -2 \frac{\text{Tr}(H - e^{-\tau} R)^2}{1 - e^{-2\tau}} \right\} dH \quad (40)$$

with

$$dH = \prod_{j=1}^M dH_{jj} \prod_{j < l}^M \prod_{k=0}^3 dH_{jl}^{(k)}. \quad (41)$$

Here $H_{jl}^{(k)}$ is the k -th component of H_{jl} . We are interested in the parametric motion of the matrix H depending on the fictitious time parameter τ .

At the initial time $\tau = 0$, this p.d.f. is reduced to

$$P(H; 0 | R) dH = \delta(H - R) dH, \quad (42)$$

so that the self-dual real quaternion matrix R gives the initial condition of the parametric motion. On the other hand, in the limit $\tau \rightarrow \infty$,

$$P(H; \infty | R) dH \propto \exp(-2 \text{Tr} H^2) dH, \quad (43)$$

which is the p.d.f. of the GSE.

Let us suppose that the elements of the initial matrix R have only the 0-th components (R is then a real symmetric matrix) and that the p.d.f. of R is

$$P_{\text{GOE}}(R) dR \propto \exp \left(-\frac{1}{2} \text{Tr} R^2 \right) dR \quad (44)$$

with

$$dR = \prod_{j=1}^M dR_{jj} \prod_{j < l}^M dR_{jl}, \quad (45)$$

which is the p.d.f. of the GOE. Then we can calculate the p.d.f. of H at a fictitious time τ as

$$\begin{aligned} P(H) dH &= \left\{ \int dR P(H; \tau | R) P_{\text{GOE}}(R) \right\} dH \\ &\propto \left[\int dR \exp \left\{ -2 \frac{\text{Tr}(H - e^{-\tau} R)^2}{1 - e^{-2\tau}} - \frac{1}{2} \text{Tr} R^2 \right\} \right] dH \\ &\propto \exp \left[-\frac{2}{1 + 3e^{-2\tau}} \sum_{j=1}^M (H_{jj}^{(0)})^2 - \frac{4}{1 + 3e^{-2\tau}} \sum_{j < l}^M (H_{jl}^{(0)})^2 \right. \\ &\quad \left. - \frac{4}{1 - e^{-2\tau}} \sum_{j < l}^M \left\{ (H_{jl}^{(1)})^2 + (H_{jl}^{(2)})^2 + (H_{jl}^{(3)})^2 \right\} \right] dH, \end{aligned} \quad (46)$$

which describes the crossover from the GOE (at $\tau = 0$) to GSE (at $\tau = \infty$). The components of the elements H_{jl} ($j \leq l$) are independently distributed according to Gaussian density functions.

If an $M \times M$ real quaternion matrix Z satisfies

$$Z \bar{Z} = \bar{Z} Z = I_M \quad (47)$$

(I_M is the $M \times M$ identity matrix), then Z is called a symplectic matrix. If the elements of a symplectic matrix U only have the 0-th elements, then U is a real orthogonal matrix. It is known that the measure dH is invariant under the symplectic transformation $H \mapsto \bar{Z} H Z$ and the measure dS is invariant under the orthogonal transformation $S \mapsto U^T S U$ (U^T is the transpose of U). It follows that the p.d.f. $P(H) dH$ in (46) is invariant under the orthogonal transformation $H \mapsto U^T H U$.

We go back to the problem of a chaotic cavity with M bound states to which two leads with N_1 and N_2 propagating modes are attached ($M \geq N = N_1 + N_2$). We are interested in the limit $M \rightarrow \infty$. Let us suppose that the $M \times M$ matrix H describing the scattering in the cavity is a random matrix distributed according to the crossover p.d.f. in (46). Then the $N \times N$ scattering matrix S is²

$$S = I_N + i2\pi W^T (H - E_F - i\pi W W^T)^{-1} W, \quad (48)$$

where E_F is the Fermi energy, and the elements of an $M \times N$ real matrix W are the coupling constants between the cavity and the leads.

Assuming that the tunnel probability of the leads are 1, we can see that the eigenvalues of $W^T W$ are all $M/(\rho\pi^2)$, where ρ is the eigenvalue density of H at the Fermi energy. Then a singular value decomposition

$$W = U D V \quad (49)$$

holds, where U and V are $M \times M$ and $N \times N$ real orthogonal matrices, respectively, and D is an $M \times N$ matrix

$$D = \frac{1}{\pi} \sqrt{\frac{M}{\rho}} \tilde{W} \quad (50)$$

with

$$\tilde{W} = \begin{pmatrix} I_N \\ O \end{pmatrix}. \quad (51)$$

Here O is an $(M - N) \times N$ matrix consisting of zero elements. When the Fermi energy E_F is set to zero (so that $\rho = \sqrt{2M}/\pi$), the scattering matrix S can be rewritten as

$$S = I_N + i\sqrt{2M}\tilde{W}^T \left(\tilde{H} - i\sqrt{\frac{M}{2}}\tilde{W}\tilde{W}^T \right)^{-1} \tilde{W}, \quad (52)$$

where

$$\tilde{H} = (U\tilde{V})^T H (U\tilde{V}) \quad (53)$$

with

$$\tilde{V} = \begin{pmatrix} V & O \\ O & I_{M-N} \end{pmatrix}. \quad (54)$$

Since $U\tilde{V}$ is a real orthogonal matrix, the $M \times M$ matrix \tilde{H} is also distributed according to the p.d.f. in (46) (with H replaced by \tilde{H}).

Thus the scattering matrix S can numerically be generated by using the Gaussian p.d.f. (46) of \tilde{H} and the relation (52). Replacing the quaternion elements of S by the 2×2 matrix representations (38), we obtain a $2N \times 2N$ matrix \tilde{S} . It is written in terms of the $2N_1 \times 2N_2$ transmission matrix \mathbf{t} as

$$\tilde{S} = \begin{pmatrix} \mathbf{r} & \mathbf{t}^\dagger \\ \mathbf{t} & \mathbf{r}' \end{pmatrix}, \quad (55)$$

where \mathbf{r} and \mathbf{r}' are the reflection matrices. Then the average conductance G can be evaluated as

$$\frac{G}{G_0} = \langle \text{Tr}(\mathbf{t}\mathbf{t}^\dagger) \rangle = \left\langle \sum_{j=2N_2+1}^{2(N_1+N_2)} \sum_{l=1}^{2N_2} |\tilde{S}_{jl}|^2 \right\rangle, \quad (56)$$

where the brackets $\langle \dots \rangle$ denote an average over the p.d.f. (46). The conductance variance $\text{Var}G$ and the shot noise P can similarly be written as

$$\frac{\text{Var}G}{G_0^2} = \left\langle \{ \text{Tr}(\mathbf{t}\mathbf{t}^\dagger) \}^2 \right\rangle - \langle \text{Tr}(\mathbf{t}\mathbf{t}^\dagger) \rangle^2 \quad (57)$$

and

$$\frac{P}{P_0} = \langle \text{Tr}(\mathbf{t}\mathbf{t}^\dagger) - \text{Tr}(\mathbf{t}\mathbf{t}^\dagger \mathbf{t}\mathbf{t}^\dagger) \rangle. \quad (58)$$

The remaining task is to find a relation between the semiclassical parameter ξ and the fictitious time τ .

Pandey analyzed hierarchical relations among the eigenvalue correlation functions of random matrices and evaluated the form factor $K(k; \tau)$ (the Fourier transform of the scaled two eigenvalue correlation function). For the random matrices obeying the crossover p.d.f. in (46), he derived a relation³⁹

$$K(k; \tau) = K(k; \infty) + \{K(k; 0) - K(k; \infty)\} e^{-4\pi^2 \rho^2 \tau k} \quad (59)$$

with $k \downarrow 0$. Here $K(k; 0)$ and $K(k; \infty)$ are the form factors of the GOE and GSE random matrices, respectively. They are known to be³⁷

$$K(k; 0) = 2k, \quad K(k; \infty) = \frac{k}{2}, \quad k \downarrow 0, \quad (60)$$

so that

$$K(k; \tau) = \frac{k}{2} (1 + 3 e^{-8M\tau k}), \quad k \downarrow 0. \quad (61)$$

On the other hand, Nagao and Saito semiclassically analyzed the form factor of a chaotic system with a weak spin-orbit interaction. They obtained a small $k > 0$ expansion up to the second order³¹

$$K(k; a) = \frac{k}{2} (1 + 3 e^{-akT_H}) + \frac{k^2}{4} \{1 + (3akT_H - 9) e^{-akT_H}\} \quad (62)$$

with $a = \gamma_{so}^2 \mathcal{D}$. Comparing (61) and (62), we arrive at a relation

$$8M\tau = aT_H. \quad (63)$$

Therefore the semiclassical parameter $\xi = aT_H/N$ is associated with the random matrix parameter τ as

$$\xi = \frac{8M}{N} \tau. \quad (64)$$

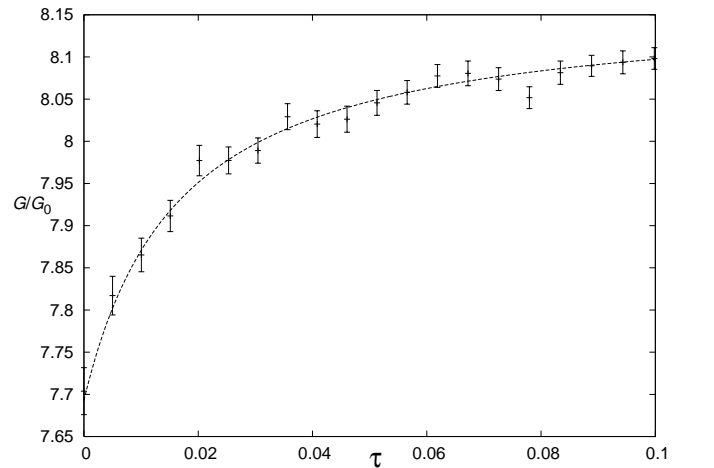


FIG. 5: A comparison of the semiclassical result (curve) and numerically calculated random matrix results (errorbars) for G/G_0 .

In Fig. 5, numerical calculations of G/G_0 at various values of τ are compared with the corresponding semiclassical predictions (25) with $\xi = 8M\tau/N$. In the numerical calculations, we set $M = 200$, $N_1 = 20$ and $N_2 = 5$. The errorbars are introduced in order to estimate the statistical errors due to the fact that the averages are calculated over only 300 samples. Note that the semiclassical formulas are truncated and hence are valid only for large N_1 and N_2 . Nevertheless we can see a fairly reasonable agreement with the numerical results.

Similar plots for $\text{Var}G/G_0^2$ and P/P_0 are also shown, respectively, in Figs. 6 and 7. The semiclassical curves are drawn by using eqs. (31) and (35). Since we again find reasonable agreements, it can be conjectured that there is an equivalence between the semiclassical method and random matrix theory.

In the case of the GOE-GUE crossover, the corresponding random matrix model can analytically be treated⁸ and the results can be compared with the semiclassical formulas²². It seems possible to apply similar techniques to the GOE-GSE crossover. For example, the diagrammatic perturbation theory is able to give the leading terms of the transport properties^{14–17}. It would be interesting to compare the semiclassical formulas with such analytical results and confirm the equivalence mentioned above.

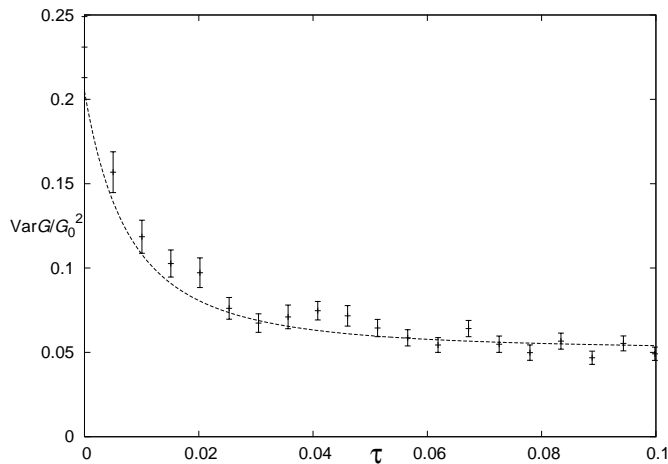


FIG. 6: A comparison of the semiclassical result (curve) and numerically calculated random matrix results (errorbars) for $\text{Var}G/G_0^2$.

V. SUMMARY

We studied a chaotic quantum transport of an electron with spin-orbit interaction in a cavity. Our approach is based on the semiclassical theory. The key ingredient of the theory is the universal statistics of the stability amplitudes. The electron diffusion in the position and momentum space is related to the escape rate, and the

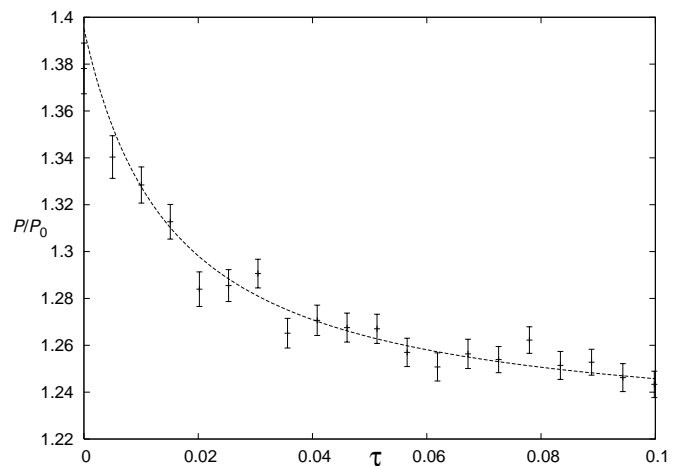


FIG. 7: A comparison of the semiclassical result (curve) and numerically calculated random matrix results (errorbars) for P/P_0 .

spin diffuses on the Bloch sphere due to the spin-orbit interaction, where the momentum variable plays the role of a stochastic magnetic field. Consequently the crossover parameter depends on the diffusion constant of the spin. The spin diffusion terms appear in a non-Abelian way along the classical trajectories. The spins along the trajectories interfere with each other, resulting in the change of the total spin. For instance, Eq. (23) has both singlet and triplet contributions. These kinds of interference effects seem to play a crucial role.

In our calculation of the physical quantities such as the average conductance, only the first several terms of the resulting expansions were worked out. Such truncated results are valid only when the channel numbers are large. In order to obtain the full expansions, a more systematic calculation of the spin diffusion terms would be necessary. Although our expansions are truncated, they are still useful in the experimental point of view, because the channel numbers can be very large. Moreover the GOE-GSE crossover can be realized when the spin-orbit interaction is controlled by applying an electric field. Therefore we believe that an experimental test of our theory is in principle possible.

In addition to the transport properties analyzed in this paper, the shot-noise variance is also an important quantity, which can be treated in an RMT framework⁴⁰. It seems possible to apply the semiclassical method to the shot-noise variance. More ambitiously, one might be able to calculate arbitrary order cumulants of the conductance and shot noise in a semiclassical framework, as the RMT approach has already made a progress in that direction⁴¹.

Our theory is valid in the case that the dwell time of the electron is much larger than the Ehrenfest time. When the Ehrenfest time is relatively large, the resulting corrections should be considered⁴². In addition, the spin diffusion mechanism might depend on the specific form of the spin-orbit coupling^{15,20}. These problems are also

interesting in experimentally realizable situations.

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Appendix A: Spin diffusion terms

In this Appendix, we present the spin diffusion terms contributing to the average conductance, conductance variance and shot noise. Here T_j and t_j are the times elapsed on the loop L_j and encounter E_j , respectively, and $a = \gamma_{\text{so}}^2 \mathcal{D}$. These spin diffusion terms were evaluated by using Mathematica.

1. Average conductance

The spin diffusion terms contributing to the average conductance (corresponding to the diagrams shown in Fig.2) are listed below.

$$\begin{aligned}
\text{(i)} : & \left\langle \left\langle \text{Tr} \left\{ \Delta_{L_1} \Delta_{E_1} \Delta_{L_2} \Delta_{E_2} \Delta_{L_3} \Delta_{E_1} \Delta_{L_4} \Delta_{E_2} \Delta_{L_5} (\Delta_{L_1} \Delta_{E_1} \Delta_{L_4} \Delta_{E_2} \Delta_{L_3} \Delta_{E_1} \Delta_{L_2} \Delta_{E_2} \Delta_{L_5})^\dagger \right\} \right\rangle \right\rangle \\
& = \left\langle \left\langle \text{Tr} \left\{ \Delta_{L_2} \Delta_{E_2} \Delta_{L_3} \Delta_{E_1} \Delta_{L_4} (\Delta_{L_2})^{-1} (\Delta_{E_1})^{-1} (\Delta_{L_3})^{-1} (\Delta_{E_2})^{-1} (\Delta_{L_4})^{-1} \right\} \right\rangle \right\rangle \\
& = \frac{1}{2} + \frac{3}{2} e^{-a(2t_1+2t_2+2T_2+2T_3)} + \frac{3}{2} e^{-a(2T_2+2T_4)} + \frac{3}{2} e^{-a(2t_1+2t_2+2T_3+2T_4)} - 3e^{-a(2t_1+2t_2+2T_2+2T_3+2T_4)}. \\
\text{(ii)} : & \left\langle \left\langle \text{Tr} \left[\Delta_{L_1} \Delta_{E_1} \Delta_{L_2} \Delta_{E_2} \Delta_{L_3} (\Delta_{E_1})^{-1} \Delta_{L_4} \Delta_{E_2} \Delta_{L_5} \left\{ \Delta_{L_1} \Delta_{E_1} (\Delta_{L_3})^{-1} (\Delta_{E_2})^{-1} (\Delta_{L_4})^{-1} \Delta_{E_1} \Delta_{L_2} \Delta_{E_2} \Delta_{L_5} \right\}^\dagger \right] \right\rangle \right\rangle \\
& = \left\langle \left\langle \text{Tr} \left\{ \Delta_{L_2} \Delta_{E_2} \Delta_{L_3} (\Delta_{E_1})^{-1} \Delta_{L_4} (\Delta_{L_2})^{-1} (\Delta_{E_1})^{-1} \Delta_{L_4} \Delta_{E_2} \Delta_{L_3} \right\} \right\rangle \right\rangle \\
& = \frac{1}{2} - \frac{3}{2} e^{-a(2t_2+2T_2+2T_3)} - \frac{3}{2} e^{-a(2t_1+2T_2+2T_4)} + \frac{3}{2} e^{-a(2t_1+2t_2+2T_3+2T_4)} + 3e^{-a(2t_1+2t_2+2T_2+2T_3+2T_4)}. \\
\text{(iii)} : & \left\langle \left\langle \text{Tr} \left[\Delta_{L_1} \Delta_{E_1} \Delta_{L_2} \Delta_{E_2} \Delta_{L_3} \Delta_{E_1} \Delta_{L_4} (\Delta_{E_2})^{-1} \Delta_{L_5} \right. \right. \right. \\
& \quad \left. \left. \times \left\{ \Delta_{L_1} \Delta_{E_1} \Delta_{L_4} (\Delta_{E_2})^{-1} (\Delta_{L_2})^{-1} (\Delta_{E_1})^{-1} (\Delta_{L_3})^{-1} (\Delta_{E_2})^{-1} \Delta_{L_5} \right\}^\dagger \right] \right\rangle \right\rangle \\
& = \left\langle \left\langle \text{Tr} \left\{ \Delta_{L_2} \Delta_{E_2} \Delta_{L_3} \Delta_{E_1} \Delta_{L_4} \Delta_{L_3} \Delta_{E_1} \Delta_{L_2} \Delta_{E_2} (\Delta_{L_4})^{-1} \right\} \right\rangle \right\rangle \\
& = \frac{1}{2} - \frac{3}{2} e^{-a(2t_1+2T_3+2T_4)} + \frac{3}{2} e^{-a(2t_1+2t_2+2T_2+2T_3)} - \frac{3}{2} e^{-a(2t_2+2T_2+2T_4)} + 3e^{-a(2t_1+2t_2+2T_2+2T_3+2T_4)}. \\
\text{(iv)} : & \left\langle \left\langle \text{Tr} \left[\Delta_{L_1} \Delta_{E_1} \Delta_{L_2} (\Delta_{E_1})^{-1} \Delta_{L_3} \Delta_{E_2} \Delta_{L_4} (\Delta_{E_2})^{-1} \Delta_{L_5} \right. \right. \right. \\
& \quad \left. \left. \times \left\{ \Delta_{L_1} \Delta_{E_1} (\Delta_{L_2})^{-1} (\Delta_{E_1})^{-1} \Delta_{L_3} \Delta_{E_2} (\Delta_{L_4})^{-1} (\Delta_{E_2})^{-1} \Delta_{L_5} \right\}^\dagger \right] \right\rangle \right\rangle \\
& = \left\langle \left\langle \text{Tr} \left\{ \Delta_{L_2} (\Delta_{E_1})^{-1} \Delta_{L_3} \Delta_{E_2} \Delta_{L_4} \Delta_{L_4} (\Delta_{E_2})^{-1} (\Delta_{L_3})^{-1} \Delta_{E_1} \Delta_{L_2} \right\} \right\rangle \right\rangle \\
& = \frac{1}{2} - \frac{3}{2} e^{-a2T_2} + \frac{9}{2} e^{-a(2T_2+2T_4)} - \frac{3}{2} e^{-a2T_4}. \\
\text{(v)} : & \left\langle \left\langle \text{Tr} \left[\Delta_{L_1} \Delta_{E_1} \Delta_{L_2} \Delta_{E_2} \Delta_{L_3} (\Delta_{E_2})^{-1} \Delta_{L_4} (\Delta_{E_1})^{-1} \Delta_{L_5} \right. \right. \right. \\
& \quad \left. \left. \times \left\{ \Delta_{L_1} \Delta_{E_1} (\Delta_{L_4})^{-1} \Delta_{E_2} \Delta_{L_3} (\Delta_{E_2})^{-1} (\Delta_{L_2})^{-1} (\Delta_{E_1})^{-1} \Delta_{L_5} \right\}^\dagger \right] \right\rangle \right\rangle \\
& = \left\langle \left\langle \text{Tr} \left\{ \Delta_{L_2} \Delta_{E_2} \Delta_{L_3} (\Delta_{E_2})^{-1} \Delta_{L_4} \Delta_{L_2} \Delta_{E_2} (\Delta_{L_3})^{-1} (\Delta_{E_2})^{-1} \Delta_{L_4} \right\} \right\rangle \right\rangle \\
& = \frac{1}{2} - \frac{3}{2} e^{-a2T_3} + \frac{3}{2} e^{-a(2T_2+2T_4)} + \frac{3}{2} e^{-a(2T_2+2T_3+2T_4)}. \\
\text{(vi)} : & \left\langle \left\langle \text{Tr} \left[\Delta_{L_1} \Delta_{E_1} \Delta_{L_2} \Delta_{E_1} \Delta_{L_3} \Delta_{E_1} \Delta_{L_4} \left\{ \Delta_{L_1} \Delta_{E_1} \Delta_{L_3} \Delta_{E_1} \Delta_{L_2} \Delta_{E_1} \Delta_{L_4} \right\}^\dagger \right] \right\rangle \right\rangle \\
& = \left\langle \left\langle \text{Tr} \left\{ \Delta_{L_2} \Delta_{E_1} \Delta_{L_3} (\Delta_{L_2})^{-1} (\Delta_{E_1})^{-1} (\Delta_{L_3})^{-1} \right\} \right\rangle \right\rangle \\
& = \frac{1}{2} + \frac{3}{2} e^{-a(2t_1+2T_2)} + \frac{3}{2} e^{-a(2t_1+2T_3)} + \frac{3}{2} e^{-a(2T_2+2T_3)} - 3e^{-a(2t_1+2T_2+2T_3)}. \\
\text{(vii)} : & \left\langle \left\langle \text{Tr} \left[\Delta_{L_1} \Delta_{E_1} \Delta_{L_2} (\Delta_{E_1})^{-1} \Delta_{L_3} \Delta_{E_1} \Delta_{L_4} \left\{ \Delta_{L_1} \Delta_{E_1} (\Delta_{L_2})^{-1} (\Delta_{E_1})^{-1} (\Delta_{L_3})^{-1} \Delta_{E_1} \Delta_{L_4} \right\}^\dagger \right] \right\rangle \right\rangle \\
& = \left\langle \left\langle \text{Tr} \left\{ \Delta_{L_2} (\Delta_{E_1})^{-1} \Delta_{L_3} \Delta_{L_3} \Delta_{E_1} \Delta_{L_2} \right\} \right\rangle \right\rangle \\
& = \frac{1}{2} - \frac{3}{2} e^{-a2T_2} + \frac{9}{2} e^{-a(2T_2+2T_3)} - \frac{3}{2} e^{-a2T_3}. \\
\text{(viii)} : & \left\langle \left\langle \text{Tr} \left[\Delta_{L_1} \Delta_{E_1} \Delta_{L_2} \Delta_{E_1} \Delta_{L_3} (\Delta_{E_1})^{-1} \Delta_{L_4} \left\{ \Delta_{L_1} \Delta_{E_1} \Delta_{L_3} (\Delta_{E_1})^{-1} (\Delta_{L_2})^{-1} (\Delta_{E_1})^{-1} \Delta_{L_4} \right\}^\dagger \right] \right\rangle \right\rangle \\
& = \left\langle \left\langle \text{Tr} \left\{ \Delta_{L_2} \Delta_{E_1} \Delta_{L_3} \Delta_{L_2} \Delta_{E_1} (\Delta_{L_3})^{-1} \right\} \right\rangle \right\rangle \\
& = \frac{1}{2} + \frac{3}{2} e^{-a(2t_1+2T_2)} + \frac{3}{2} e^{-a(2t_1+2T_2+2T_3)} - \frac{3}{2} e^{-a2T_3}. \\
\text{(ix)} : & \left\langle \left\langle \text{Tr} \left[\Delta_{L_1} \Delta_{E_1} \Delta_{L_2} (\Delta_{E_1})^{-1} \Delta_{L_3} (\Delta_{E_1})^{-1} \Delta_{L_4} \left\{ \Delta_{L_1} \Delta_{E_1} (\Delta_{L_3})^{-1} \Delta_{E_1} \Delta_{L_2} (\Delta_{E_1})^{-1} \Delta_{L_4} \right\}^\dagger \right] \right\rangle \right\rangle \\
& = \left\langle \left\langle \text{Tr} \left\{ \Delta_{L_2} (\Delta_{E_1})^{-1} \Delta_{L_3} (\Delta_{L_2})^{-1} (\Delta_{E_1})^{-1} \Delta_{L_3} \right\} \right\rangle \right\rangle \\
& = \frac{1}{2} - \frac{3}{2} e^{-a2T_2} + \frac{3}{2} e^{-a(2t_1+2T_3)} + \frac{3}{2} e^{-a(2t_1+2T_2+2T_3)}.
\end{aligned}$$

2. Conductance variance

The spin diffusion terms contributing to the conductance variance (corresponding to the diagrams shown in Fig.3) are listed below.

$$\begin{aligned}
\text{(i)} : & \left\langle \left\langle \text{Tr} \left[\Delta_{L_1} \Delta_{E_1} \Delta_{L_2} (\Delta_{E_1})^{-1} \Delta_{L_3} \left\{ \Delta_{L_1} \Delta_{E_1} (\Delta_{L_2})^{-1} (\Delta_{E_1})^{-1} \Delta_{L_3} \right\}^\dagger \right] \text{Tr} \left\{ \Delta_{L_4} (\Delta_{L_4})^\dagger \right\} \right\rangle \right\rangle \\
& = \left\langle \left\langle 2 \text{Tr} \left\{ (\Delta_{L_2})^2 \right\} \right\rangle \right\rangle = -2 + 6e^{-a2T_2}. \\
\text{(ii)} : & \left\langle \left\langle \text{Tr} \left\{ \Delta_{L_1} \Delta_{E_1} \Delta_{L_2} \Delta_{E_2} \Delta_{L_3} (\Delta_{L_1} \Delta_{E_1} \Delta_{L_5} \Delta_{E_2} \Delta_{L_3})^\dagger \right\} \text{Tr} \left\{ \Delta_{L_4} \Delta_{E_1} \Delta_{L_5} \Delta_{E_2} \Delta_{L_6} (\Delta_{L_4} \Delta_{E_1} \Delta_{L_2} \Delta_{E_2} \Delta_{L_6})^\dagger \right\} \right\rangle \right\rangle \\
& = \left\langle \left\langle \left| \text{Tr} \left\{ \Delta_{L_2} (\Delta_{L_5})^\dagger \right\} \right|^2 \right\rangle \right\rangle = 1 + 3e^{-a(2T_2+2T_5)}. \\
\text{(iii)} : & \left\langle \left\langle \text{Tr} \left\{ \Delta_{L_1} \Delta_{E_1} \Delta_{L_2} \Delta_{E_2} \Delta_{L_3} \Delta_{E_1} \Delta_{L_4} (\Delta_{L_1} \Delta_{E_1} \Delta_{L_4})^\dagger \right\} \text{Tr} \left\{ \Delta_{L_5} \Delta_{E_2} \Delta_{L_6} (\Delta_{L_5} \Delta_{E_2} \Delta_{L_3} \Delta_{E_1} \Delta_{L_2} \Delta_{E_2} \Delta_{L_6})^\dagger \right\} \right\rangle \right\rangle \\
& = \left\langle \left\langle \left| \text{Tr} (\Delta_{L_2} \Delta_{E_2} \Delta_{L_3} \Delta_{E_1}) \right|^2 \right\rangle \right\rangle = 1 + 3e^{-a(2t_1+2t_2+2T_2+2T_3)}. \\
\text{(iv)} : & \left\langle \left\langle \text{Tr} \left\{ \Delta_{L_1} \Delta_{E_1} \Delta_{L_2} \Delta_{E_1} \Delta_{L_3} (\Delta_{L_1} \Delta_{E_1} \Delta_{L_3})^\dagger \right\} \text{Tr} \left\{ \Delta_{L_4} \Delta_{E_1} \Delta_{L_5} (\Delta_{L_4} \Delta_{E_1} \Delta_{L_2} \Delta_{E_1} \Delta_{L_5})^\dagger \right\} \right\rangle \right\rangle \\
& = \left\langle \left\langle \left| \text{Tr} (\Delta_{L_2} \Delta_{E_1}) \right|^2 \right\rangle \right\rangle = 1 + 3e^{-a(2t_1+2T_2)}. \\
\text{(v)} : & \left\langle \left\langle \text{Tr} \left[\Delta_{L_1} \Delta_{E_1} \Delta_{L_2} \Delta_{E_2} \Delta_{L_3} \left\{ \Delta_{L_1} \Delta_{E_1} (\Delta_{L_5})^{-1} \Delta_{E_2} \Delta_{L_3} \right\}^\dagger \right] \right. \right. \\
& \quad \times \left. \left. \text{Tr} \left[\Delta_{L_4} (\Delta_{E_2})^{-1} \Delta_{L_5} (\Delta_{E_1})^{-1} \Delta_{L_6} \left\{ \Delta_{L_4} (\Delta_{E_2})^{-1} (\Delta_{L_2})^{-1} (\Delta_{E_1})^{-1} \Delta_{L_6} \right\}^\dagger \right] \right] \right\rangle \right\rangle \\
& = \left\langle \left\langle \left\{ \text{Tr} (\Delta_{L_2} \Delta_{L_5}) \right\}^2 \right\rangle \right\rangle = 1 + 3e^{-a(2T_2+2T_5)}. \\
\text{(vi)} : & \left\langle \left\langle \text{Tr} \left\{ \Delta_{L_1} \Delta_{E_1} \Delta_{L_2} \Delta_{E_2} \Delta_{L_3} \Delta_{E_1} \Delta_{L_4} (\Delta_{L_1} \Delta_{E_1} \Delta_{L_4})^\dagger \right\} \right. \right. \\
& \quad \times \left. \left. \text{Tr} \left[\Delta_{L_5} (\Delta_{E_2})^{-1} \Delta_{L_6} \left\{ \Delta_{L_5} (\Delta_{E_2})^{-1} (\Delta_{L_2})^{-1} (\Delta_{E_1})^{-1} (\Delta_{L_3})^{-1} (\Delta_{E_2})^{-1} \Delta_{L_6} \right\}^\dagger \right] \right] \right\rangle \right\rangle \\
& = \left\langle \left\langle \left\{ \text{Tr} (\Delta_{L_2} \Delta_{E_2} \Delta_{L_3} \Delta_{E_1}) \right\}^2 \right\rangle \right\rangle = 1 + 3e^{-a(2t_1+2t_2+2T_2+2T_3)}. \\
\text{(vii)} : & \left\langle \left\langle \text{Tr} \left\{ \Delta_{L_1} \Delta_{E_1} \Delta_{L_2} \Delta_{E_1} \Delta_{L_3} (\Delta_{L_1} \Delta_{E_1} \Delta_{L_3})^\dagger \right\} \text{Tr} \left[\Delta_{L_4} (\Delta_{E_1})^{-1} \Delta_{L_5} \left\{ \Delta_{L_4} (\Delta_{E_1})^{-1} (\Delta_{L_2})^{-1} (\Delta_{E_1})^{-1} \Delta_{L_5} \right\}^\dagger \right] \right\rangle \right\rangle \\
& = \left\langle \left\langle \left\{ \text{Tr} (\Delta_{L_2} \Delta_{E_1}) \right\}^2 \right\rangle \right\rangle = 1 + 3e^{-a(2t_1+2T_2)}. \\
\text{(viii)} : & \left\langle \left\langle \text{Tr} \left\{ \Delta_{L_1} \Delta_{E_1} \Delta_{L_2} (\Delta_{L_1} \Delta_{E_1} \Delta_{L_4})^\dagger \right\} \text{Tr} \left\{ \Delta_{L_3} \Delta_{E_1} \Delta_{L_4} (\Delta_{L_3} \Delta_{E_1} \Delta_{L_2})^\dagger \right\} \right\rangle \right\rangle \\
& = \left\langle \left\langle \left| \text{Tr} (\Delta_{L_2} \Delta_{L_4}^\dagger) \right|^2 \right\rangle \right\rangle = 1 + 3e^{-a(2T_2+2T_4)}.
\end{aligned}$$

3. Shot noise

The spin diffusion terms contributing to the shot noise (corresponding to the diagrams shown in Fig.4) are listed below.

$$\begin{aligned}
\text{(i)} : & \left\langle \left\langle \text{Tr} \left[\Delta_{L_1} \Delta_{E_1} \Delta_{L_2} (\Delta_{L_3} \Delta_{E_1} \Delta_{L_2})^\dagger \Delta_{L_3} \Delta_{E_1} \Delta_{L_4} \Delta_{E_2} \Delta_{L_5} (\Delta_{E_2})^{-1} \Delta_{L_6} \right. \right. \right. \\
& \quad \times \left. \left. \left\{ \Delta_{L_1} \Delta_{E_1} \Delta_{L_4} \Delta_{E_2} (\Delta_{L_5})^{-1} (\Delta_{E_2})^{-1} \Delta_{L_6} \right\}^\dagger \right] \right\rangle \right\rangle \\
& = \left\langle \left\langle \text{Tr} \left\{ (\Delta_{L_5})^2 \right\} \right\rangle \right\rangle = -1 + 3e^{-2aT_5}.
\end{aligned}$$

$$\begin{aligned}
\text{(ii)} : & \left\langle\left\langle \text{Tr} \left[\Delta_{L_1} \Delta_{E_1} \Delta_{L_2} \left\{ \Delta_{L_5} \Delta_{E_2} (\Delta_{L_3})^{-1} \Delta_{E_1} \Delta_{L_2} \right\}^\dagger \Delta_{L_5} \Delta_{E_2} \Delta_{L_4} (\Delta_{E_1})^{-1} \Delta_{L_3} (\Delta_{E_2})^{-1} \Delta_{L_6} \right. \right. \right. \\
& \times \left. \left. \left\{ \Delta_{L_1} \Delta_{E_1} (\Delta_{L_4})^{-1} (\Delta_{E_2})^{-1} \Delta_{L_6} \right\}^\dagger \right] \right\rangle\right\rangle \\
& = \left\langle\left\langle \text{Tr} \left\{ (\Delta_{E_1}^\dagger \Delta_{L_3} \Delta_{L_4})^2 \right\} \right\rangle\right\rangle = -1 + 3e^{-2a(t_1+T_3+T_4)}.
\end{aligned}$$

$$\begin{aligned}
\text{(iii)} : & \left\langle\left\langle \text{Tr} \left[\Delta_{L_1} \Delta_{E_1} \Delta_{L_2} (\Delta_{L_3} \Delta_{E_1} \Delta_{L_2})^\dagger \Delta_{L_3} \Delta_{E_1} \Delta_{L_4} (\Delta_{E_1})^{-1} \Delta_{L_5} \left\{ \Delta_{L_1} \Delta_{E_1} (\Delta_{L_4})^{-1} (\Delta_{E_1})^{-1} \Delta_{L_5} \right\}^\dagger \right] \right\rangle\right\rangle \\
& = \left\langle\left\langle \text{Tr} \left\{ (\Delta_{L_4})^2 \right\} \right\rangle\right\rangle = -1 + 3e^{-2aT_4}.
\end{aligned}$$
